The general $U_q(sl(2))$ invariant XXZ integrable quantum spin chain

1991 J. Phys. A: Math. Gen. 24 L435

(http://iopscience.iop.org/0305-4470/24/8/009)

J. Phys. A: Math. Gen. 24 (1991) L435–L439. Printed in the UK

LETTER TO THE EDITOR

The general $U_q[sl(2)]$ invariant *XXZ* integrable quantum spin chain

P P Kulish† and E K Sklyanin‡§‖

† Leningrad Branch of Steklov Mathematical Institute, Fontanka 27, Leningrad 191011, USSR

‡ Research Institute for Theoretical Physics, University of Helsinki, Siltavuorenpenger 20C, 00170 Helsinki 17, Finland

Received 18 February 1991

Abstract. The boundary conditions are constructed for the general *XXZ* quantum integrable spin chain such that the resulting system is invariant under the quantum algebra $U_q[sl(2)]$ action. The result generalizes that due to Pasquier and Saleur, obtained for spin-$\frac{1}{2}$ and spin-1.

The open *XXZ* quantum spin-$\frac{1}{2}$ chain characterized by the three-parametric Hamiltonian

$$\mathscr{H}_{qp\pm} = \sum_{n=1}^{N-1} \left(\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \frac{q+q^{-1}}{2} \sigma_n^z \sigma_{n+1}^z \right) + \frac{q-q^{-1}}{2} (p_- \sigma_1^z + p_+ \sigma_N^z) \tag{1}$$

expressed in terms of the Pauli matrices σ is well known to be exactly soluble. The spectrum and the eigenfunctions of $\mathscr{H}_{qp\pm}$ can be found via Bethe ansatz technique (Gaudin 1978, Alcaraz *et al* 1987, Sklyanin 1988). Note that $\mathscr{H}_{qp\pm}$ commutes with the operator

$$H = \tfrac{1}{2}(\sigma_1^z + \ldots + \sigma_N^z). \tag{2}$$

Recently, Pasquier and Saleur (1990) noticed that if the relation

$$p_\pm = \mp 1 \tag{3}$$

is imposed the resulting one-parametric Hamiltonian $\mathscr{H}_q$ commutes also with the operators

$$X^\pm = \sum_{n=1}^{N} q^{(\sigma_1^z + \ldots + \sigma_{n-1}^z)/2} \frac{\sigma_n^x \pm i\sigma_n^y}{2} q^{-(\sigma_{n+1}^z + \ldots + \sigma_N^z)/2} \tag{4}$$

which, taken together with H, form a representation of the quantum algebra $U_q[sl(2)]$ (Jimbo 1985, Faddeev *et al* 1988)

$$HX^\pm - X^\pm H = \pm X^\pm \qquad X^+X^- - X^-X^+ = [2H]_q \tag{5}$$

where $[x]_q = (q^x - q^{-x})/(q - q^{-1})$. The same result is obtained also for the spin-1 chain (Pasquier and Saleur 1990, Mezincescu *et al* 1990).

§ Permanent address: Leningrad Branch of Steklov Mathematical Institute, Fontanka 27, Leningrad 191011, USSR.

‖ bitnet: SKLYANIN@FINUHCB

In the present letter we show how such an additional global symmetry can be understood in terms of the quantum inverse scattering method (QISM) (Faddeev 1984). Our treatment is based on the general R matrix approach to the boundary conditions for integrable quantum models developed by Sklyanin (1988). We prove that for a generic quantum spin system having the same R matrix as the XXZ chain there exists such a boundary condition that the system is $U_q[\mathrm{sl}(2)]$ invariant. Note that $U_q[\mathrm{sl}(2)]$ invariance is shared not only by the simplest Hamiltonians like $\mathcal{H}_q$ but also by the whole family of commutative quantum integrals of motion. Some results of the present letter were announced by Kulish and Sklyanin (1991).

Consider the associative algebra T defined by the set of generators $T_{ij}(\lambda)$, $i, j \in \{1, 2\}$, $\lambda \in \mathbb{C}$ forming a 2×2 matrix $T(\lambda)$ and by the quadratic relations

$$R(\lambda/\mu)\overset{1}{T}(\lambda)\overset{2}{T}(\mu) = \overset{2}{T}(\mu)\overset{1}{T}(\lambda)R(\lambda/\mu) \tag{6}$$

where $\overset{1}{T} = T\otimes I$, $\overset{2}{T} = I\otimes T$, I is the unit 2×2 matrix and $R(\lambda/\mu)$ is the usual R matrix of the XXZ spin chain

$$R(\lambda/\mu) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix} \tag{7}$$

$$a = \lambda\mu^{-1}q - \lambda^{-1}\mu q^{-1} \qquad b = \lambda\mu^{-1} - \lambda^{-1}\mu \qquad c = q - q^{-1}$$

depending on the parameter q.

To any given representation of T there corresponds some integrable quantum system whose commuting integrals of motion (in case of the periodic boundary conditions) are given by the matrix trace $t(\lambda) = \mathrm{tr}\, T(\lambda) = T_{ii}(\lambda)$. This fact lies in the basis of QISM (Faddeev 1984).

Consider now the general non-homogeneous integrable quantum XXZ chain which corresponds to the following representation of T

$$T(\lambda) = L_N(\lambda d_N^{-1}) \ldots L_2(\lambda d_2^{-1})L_1(\lambda d_1^{-1}) \tag{8}$$

$$L_n(\lambda) = \begin{pmatrix} \lambda q^{H_n} - \lambda^{-1}q^{-H_n} & (q-q^{-1})X_n^- \\ (q-q^{-1})X_n^+ & \lambda q^{-H_n} - \lambda^{-1}q^{H_n} \end{pmatrix} \tag{9}$$

where the local operators H_n, $X_n^{\pm}$ commute for $n_1 \neq n_2$ and for fixed n form irreducible representations of $U_q[\mathrm{sl}(2)]$ (5) characterized by the values of the Casimir operators

$$\frac{q+q^{-1}}{2}[H_n]_q^2 + \tfrac{1}{2}(X_n^+X_n^- + X_n^-X_n^+) = [s_n]_q[s_n+1]_q. \tag{10}$$

To sum up, the representation considered depends on $2N$ parameters: q-spins $\{s_n\}_{n=1}^N$ and shifts $\{d_n\}_{n=1}^N$. We do not discuss here the conjugation properties of H_n, $X_n^{\pm}$ since our approach is quite general and does not depend on any particular real form of $U_q[\mathrm{sl}(2)]$.

No w we shall extract from $T(\lambda)$ the operators H, $X^{\pm}$ defining the action of $U_q[\mathrm{sl}(2)]$ in the representation space of T. Let us define the matrices

$$U(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \qquad T_{\pm} = \lim_{\ln|\lambda|\to\pm\infty} \lambda^{\mp N}U^{-1}(\lambda)T(\lambda)U(\lambda). \tag{11}$$

Using (6), (7) and (11) one finds easily the commutation relations between the entries of $T_\pm$ and $T(\mu)$

$$R_\pm(\mu)\overset{1}{T}_\pm\overset{2}{T}(\mu)=\overset{2}{T}(\mu)\overset{1}{T}_\pm R_\pm(\mu) \tag{12}$$

$$R_\pm\overset{1}{T}_\pm\overset{2}{T}_\varepsilon=\overset{2}{T}_\varepsilon\overset{1}{T}_\pm R_\pm \qquad \forall\varepsilon\in\{+,-\} \tag{13}$$

$$R_\pm(\mu)=\lim_{\ln|\lambda|\to\pm\infty}\lambda^{\mp1}\mu^{\pm1}\overset{1}{U}^{-1}(\lambda)R_\pm(\lambda/\mu)\overset{1}{U}(\lambda)=\overset{2}{U}(\mu)R_\pm\overset{2}{U}^{-1}(\mu) \tag{14}$$

$$R_+=\begin{pmatrix} q & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & q-q^{-1} & 1 & 0\\ 0 & 0 & 0 & q\end{pmatrix} \qquad R_-=-(R_+^{-1})^t. \tag{15}$$

Note now that the commutation relations (13) are none other but the defining relations for $U_q[sl(2)]$ written in the R matrix form (Faddeev *et al* 1989, Takhtajan 1990). More explicitly,

$$T_+=d^{-1}\begin{pmatrix} q^H & 0\\ (q-q^{-1})X^+ & q^{-H}\end{pmatrix} \qquad T_-=(-1)^N d\begin{pmatrix} q^{-H} & -(q-q^{-1})X^-\\ 0 & q^H\end{pmatrix} \tag{16}$$

where $d=d_1d_2\dots d_N$ and the operators H, $X^\pm$

$$H=H_1+H_2+\dots+H_N \qquad X^\pm=\sum_{n=1}^{N} d_n^{\pm1}q^{H_1+\dots+H_{n-1}}X_n^\pm q^{-H_{n+1}-\dots-H_N} \tag{17}$$

satisfy the $U_q[sl(2)]$ commutation relations (5).

In order to obtain from $T(\lambda)$ an open chain with appropriate boundary conditions on each end we shall use the general recipe proposed by Sklyanin (1988). The Hamiltonians of the integrable open XXZ chain are generated by the commuting family.

$$\tau(\mu)=\operatorname{tr}K_+(\mu q^{1/2})T(\mu)K_-(\mu q^{-1/2})\tilde{T}(\mu^{-1}) \tag{18}$$

where $\tilde{T}(\mu^{-1})=\sigma_2T^t(\mu^{-1})\sigma_2$ and the matrices $K_\pm(\mu)$ specifying the boundary conditions on each end of the chain are defined as

$$K_\pm(\mu)=\frac{\mu}{2}\begin{pmatrix} 1+p_\pm & 0\\ 0 & 1-p_\pm\end{pmatrix}+\frac{\mu^{-1}}{2}\begin{pmatrix} 1-p_\pm & 0\\ 0 & 1+p_\pm\end{pmatrix} \tag{19}$$

and satisfy the relation

$$R(\lambda/\mu)\overset{1}{K}_\pm(\lambda)R(\lambda\mu)\overset{2}{K}_\pm(\mu)=\overset{2}{K}_\pm(\mu)R(\lambda\mu)\overset{1}{K}_\pm(\lambda)R(\lambda/\mu).$$

Theorem. If the condition (3) is satisfied, so that the matrices $K_\pm(\mu)$ take the form

$$K_\pm(\mu)=\begin{pmatrix} \mu^{\mp1} & 0\\ 0 & \mu^{\pm1}\end{pmatrix} \tag{20}$$

then the integrals of motion $\tau(\mu)$ (18) commute with the generators H, $X^\pm$ (17) of the quantum algebra $U_q[sl(2)]$, or

$$[T_\pm,\tau(\mu)]=0 \qquad \forall\mu. \tag{21}$$

The proof of the theorem uses several identities listed below. First, note that the relation (12) and the fact that $\tilde{T}(\mu)$ is proportional to $T^{-1}(\mu q)$ up to a scalar factor (see Sklyanin 1988) imply the identity

$$\overset{1}{T}_{\pm} R_{\pm}(\mu^{-1}q)\overset{2}{\tilde{T}}(\mu^{-1}) = \overset{2}{\tilde{T}}(\mu^{-1})R_{\pm}(\mu^{-1}q)\overset{1}{T}_{\pm}. \tag{22}$$

It is also easy to verify that the matrices $K_{\pm}(\lambda)$ (20) and $R_{\pm}(\mu)$ (14) satisfy the relations

$$\overset{2}{K}{}^{\pm1}_{\pm}(\lambda)R_{\varepsilon}(\mu)\overset{2}{K}{}^{\mp1}_{\pm}(\lambda) = R_{\varepsilon}(\mu\lambda^{-2}) \qquad \forall\varepsilon\in\{+,-\}. \tag{23}$$

The last identity to be mentioned

$$\mathrm{tr}_2\, R_{\mp}^{-1}(\lambda)\overset{2}{Z}R_{\pm}(\lambda q^2) = \overset{1}{I}\cdot \mathrm{tr}\, Z \tag{24}$$

holds for any 2×2 matrix Z and for all λ. The simplest way to verify (24) is to make the matrix transposition for the second space in $\mathbb{C}^2\otimes\mathbb{C}^2$

$$\mathrm{tr}_2\, R_{\mp}^{-1}(\lambda)\overset{2}{Z}R_{\pm}(\lambda q^2) = \mathrm{tr}_2(R_{\mp}^{-1})^{t_2}(\lambda)R_{\pm}^{t_2}(\lambda q^2)\overset{2}{Z}{}^{t}$$

and to apply the identity $(R_{\mp}^{-1})^{t_2}(\lambda)R_{\pm}^{t_2}(\lambda q^2) = I\otimes I$ which is a remnant of the unitarity and crossing properties of R matrix (7), see (Sklyanin 1988).

Now, in order to prove the identity (21) consider the product $T_{\pm}\tau(\mu)$. Using (18) and the fact that $K_{+}(\mu)$ is a c-number matrix one obtains

$$\begin{aligned}\overset{1}{T}_{\pm}\tau(\mu) &= \mathrm{tr}_2\, \overset{1}{T}_{\pm}\overset{2}{K}_{+}(\mu q^{1/2})\overset{2}{T}(\mu)\overset{2}{K}_{-}(\mu q^{-1/2})\overset{2}{\tilde{T}}(\mu^{-1})\\ &= \mathrm{tr}_2\, \overset{2}{K}_{+}(\mu q^{1/2})\overset{1}{T}_{\pm}\overset{2}{T}(\mu)\overset{2}{K}_{-}(\mu q^{-1/2})\overset{2}{\tilde{T}}(\mu^{-1}) = \ldots.\end{aligned}$$

Then one uses consecutively the identities (12), (23) and (22)

$$\begin{aligned}\ldots &= \mathrm{tr}_2\, \overset{2}{K}_{+}(\mu q^{1/2})R_{\mp}^{-1}(\mu)\overset{2}{T}(\mu)\overset{1}{T}_{\pm}R_{\pm}(\mu)\overset{2}{K}_{-}(\mu q^{-1/2})\overset{2}{\tilde{T}}(\mu^{-1})\\ &= \mathrm{tr}_2\, R_{\mp}^{-1}(\mu^{-1}q^{-1})\overset{2}{K}_{+}(\mu q^{1/2})\overset{2}{T}(\mu)\overset{1}{T}_{\pm}\overset{2}{K}_{-}(\mu q^{-1/2})R_{\pm}(\mu^{-1}q)\overset{2}{\tilde{T}}(\mu^{-1})\\ &= \mathrm{tr}_2\, R_{\mp}^{-1}(\mu^{-1}q^{-1})\overset{2}{K}_{+}(\mu q^{1/2})\overset{2}{T}(\mu)\overset{2}{K}_{-}(\mu q^{-1/2})\overset{1}{T}_{\pm}R_{\pm}(\mu^{-1}q)\overset{2}{\tilde{T}}(\mu^{-1})\\ &= \mathrm{tr}_2(R_{\mp}^{-1}(\mu^{-1}q^{-1})\overset{2}{K}_{+}(\mu q^{1/2})\overset{2}{T}(\mu)\overset{2}{K}_{-}(\mu q^{-1/2})\overset{2}{\tilde{T}}(\mu^{-1})R_{\pm}(\mu^{-1}q))\overset{1}{T}_{\pm}.\end{aligned} \tag{25}$$

Finally, using (24) for $Z = K_{+}(\mu q^{1/2})T(\mu)K(\mu q^{-1/2})\tilde{T}(\mu^{-1})$ and $\lambda = \mu^{-1}q^{-1}$ one transforms (25) into

$$\mathrm{tr}_2(\overset{2}{K}_{+}(\mu q^{1/2})\overset{2}{T}(\mu)\overset{2}{K}_{-}(\mu q^{-1/2})\overset{2}{\tilde{T}}(\mu^{-1}))\overset{1}{T}_{\pm} = \tau(\mu)\overset{1}{T}_{\pm}$$

which completes the proof of the theorem.

In case when H_n, $X_n^{\pm}$ form finite-dimensional irreducible highest weight representation of $\mathrm{U}_q[\mathrm{sl}(2)]$ (Jimbo 1985, Sklyanin 1983) the spectrum of $\tau(\mu)$ can be determined by means of the algebraic Bethe ansatz. It suffices to apply the general formulae given in (Sklyanin 1988) for the case of the matrices $K_\pm$ given by (20). The result is formulated most conveniently using the variables $u=\ln\mu$, $\eta=\ln q$, $\delta_n=\ln d_n$. The eigenvalues of $\tau(u)$

$$\tau(u)=\frac{\sinh(2u+\eta)}{\sinh 2u}\delta_+(u)\delta_-(-u)\prod_{m=1}^{M}\frac{\sinh(u-v_m-\eta)\sinh(u+v_m-\eta)}{\sinh(u-v_m)\sinh(u+v_m)}$$
$$+\frac{\sinh(2u-\eta)}{\sinh 2u}\delta_+(-u)\delta_-(u)$$
$$\times\prod_{m=1}^{M}\frac{\sinh(u-v_m+\eta)\sinh(u+v_m+\eta)}{\sinh(u-v_m)\sinh(u+v_m)} \tag{26}$$

are parametrized by the solutions $(v_1,\ldots,v_m)$ of the Bethe equations

$$\frac{\delta_+(v_m)\delta_-(-v_m)}{\delta_+(-v_m)\delta_-(v_m)}=\prod_{\substack{k=1\\k\neq m}}^{M}\frac{\sinh(v_m-v_k+\eta)\sinh(v_m+v_k+\eta)}{\sinh(v_m-v_k-\eta)\sinh(v_m+v_k-\eta)} \tag{27}$$

where $\delta_\pm(u)=\Pi_{n=1}^{N}\sinh(u-\delta_n\pm s_n\eta)$.

The results established in the present letter allows one to construct local two-point $U_q[sl(2)]$ invariant integrable Hamiltonians like (1) for homogeneous ($\delta_n\equiv 0$, $s_n\equiv s$) spin chains. It can be done with a proper generalization of the fusion procedure described for spin-1 case by Mezincescu *et al* (1990). It would be interesting to study the critical properties of these Hamiltonians, see (Alcaraz *et al* 1987) for spin-$\frac{1}{2}$ and (Martins 1990) for spin-1 cases. Another interesting problem is possible generalization of our results to other quantum algebras, for example $U_q[sl(N)]$.

One of us (ES) acknowledges the hospitality extended to him at Helsinki University. This work was supported in part by the Finnish Academy of Sciences.

References

Alcaraz F C, Barber M N, Batchelor M T, Baxter R J and Quispel G R W 1987 *J. Phys. A: Math. Gen.* **20** 6397–409

Faddeev L D 1984 *Les Houches 1982* ed J-B Zuber and R Stora (Amsterdam: North-Holland) pp 561–608

Faddeev L D, Reshetikhin N Yu and Takhtajan L A 1988 *Algebraic Analysis* vol 1, ed M Kashiwara and T Kawai (New York: Academic Press) pp 129–40

Gaudin M 1978 *Phys. Rev.* A **4** 386–94

Jimbo M 1985 *Lett. Math. Phys.* **10** 63–9

Kulish P P and Sklyanin E K 1991 *Proc. Euler Int. Math. Inst., 1st Semester: Quantum Groups, Autumn 1990* ed P P Kulish (Berlin: Springer) to be published

Martins M J 1990 *Phys. Lett.* **151A** 519–22

Mezincescu L, Nepomechie R I and Rittenberg V R 1990 *Phys. Lett.* **147A** 70–8

Pasquier V and Saleur H 1990 *Nucl. Phys.* B **330** 523–56

Sklyanin E K 1983 *Funct. Anal. Appl.* **17** 273–84

—— 1988 *J. Phys. A: Math. Gen.* **21** 2375–89

Takhtajan L A 1990 *Introduction to Quantum Groups and Integrable Massive Models of Quantum Field Theory* ed M-L Ge and B-H Zhao (Singapore: World Scientific) pp 69–197